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Probability Distributions and Coherent States of B_r, C_r and D_r Algebras

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Abstract

A new approach to probability theory based on quantum mechanical and Lie algebraic ideas is proposed and developed. The underlying fact is the observation that the coherent states of the Heisenberg-Weyl, $su(2)$, $su(r+1)$, $su(1,1)$ and $su(r,1)$ algebras in certain symmetric (bosonic) representations give the "probability amplitudes" (or the "square roots") of the well-known Poisson, binomial, multinomial, negative binomial and negative multinomial distributions in probability theory. New probability distributions are derived based on coherent states of the classical algebras B_r, C_r and D_r in symmetric representations. These new probability distributions are simple generalisation of the multinomial distributions with some added new features reflecting the quantum and Lie algebraic construction. As byproducts, simple proofs and interpretation of addition theorems of Hermite polynomials are obtained from the 'coordinate' representation of the (negative) multinomial states. In other words, these addition theorems are higher rank counterparts of the well-known generating function of Hermite polynomials, which is essentially the 'coordinate' representation of the ordinary (Heisenberg-Weyl) coherent state.

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1 Introduction

Quantum theory is one of the greatest achievements in twentieth century physics. It has changed the fundamental structure of physics, material science and also influenced various disciplines, in particular biological (genetic) science and philosophy. Quantum theory dictates that at the microscopic level nature is not governed by causal laws typically exemplified by the Newtonian equation of motion but by probabilistic laws. The fundamental ingredient of quantum theory is, however, not the probability itself but the probability amplitude which obeys a certain equation of motion and the square of which gives appropriate probabilities.

In the present paper we report on an attempt to apply *quantum theory ideas* to *probability theory itself*. This, we believe, will provide new perspectives on probability theory and hopefully will enrich the long-established and rather mature science. The first step would be to associate certain "probability amplitudes" to some typical probability distributions of classical probability theory. In a broader perspective, this problem belongs to the paradigm of "square roots". The Dirac equation is obtained as a "square root" of the Klein-Gordon equation. The creation and annihilation operators can be considered as "square roots" of the harmonic oscillator hamiltonian. Of course such a "square root" can never be unique. It depends on the formulation. It turns out that the '*coherent states*' [1, 2, 3, 4] in quantum optics and the so-called 'generalised coherent states'[1][5, 6] associated with various Lie algebras could be identified as certain "probability amplitudes". For example, the coherent states associated with the Heisenberg-Weyl algebra, $su(2)$ [7, 8], $su(r+1)$ [9, 10] and $su(1,1)$ [5, 9, 11, 12, 13] $su(r,1)$ [13] algebras in totally symmetric (bosonic) representations could well be interpreted as "probability amplitudes" for the Poisson, binomial, multinomial and negative binomial, negative multinomial distributions in probability theory, respectively [13, 14]. This also means, in turn, that these typical discrete *probability distributions are characterised in terms of Lie algebras (groups) and their representations.* The relationship between the Poisson distribution and the ordinary coherent states is well-known and that of the binomial distribution and the $su(2)$ coherent states is also known, but to a lesser degree. The characterisation of the negative binomial (multinomial) distributions by Lie-algebra representations, which is believed to be new, has been reported in our previous work [13, 14].

The second step is to extract useful information (predictions) from the characterisation "probability amplitudes = coherent states". One would naturally ask 'what would be the probability distributions associated with the other Lie algebras and/or other representa-

[1] In this paper we call them simply coherent states.

tions?' In the present paper we mainly address the problems in this step. We choose the classical Lie algebras, B_r, C_r and D_r in Cartan notation (or $so(2r+1)$, $sp(2r)$ and $so(2r)$ algebra, respectively) and construct the coherent states in the totally symmetric (bosonic) representations. This gives rise to new probability distributions, to be denoted as B_r multinomial distributions, etc. One reason for choosing the symmetric representations is that they are supposed to give closest analogs of the classical probability distributions, like the multinomial distribution. Another reason is the relative ease of the calculation and presentation.

The third step would be to discuss the time evolution (stochastic process) based not on the probability itself but on the "probability amplitude" in the spirit of quantum theory [15]. This would be the subject of our future publication.

This paper is organised as follows. In section two we explain the basic idea of introducing the "probability amplitude" by taking the simplest and well-known example of the Poisson distribution and derive the ordinary coherent state. This section is meant for wider readership. In section three we discuss the "probability amplitudes" for the binomial and multinomial distributions, the coherent states of A_1 ($su(2)$) and A_r ($su(r+1)$) algebras in a slightly different way from our previous work [14]. The representation theory aspects of these algebras are emphasised in order to facilitate the transition to the other algebras treated in later sections. As new material in this section we discuss the x (coordinate) representation of these coherent states. Based on new expressions of the A_1 and A_r coherent states, which have straightforward interpretations of "probability amplitudes" for the binomial and multinomial distributions, we obtain a simple (quantum theoretical) proof and interpretation of addition theorems of the Hermite polynomials describing the number states of harmonic oscillators. This is analogous to the well-known fact that the coordinate representation of the coherent state of the Heisenberg-Weyl group gives the generating function of Hermite polynomials. In sections four, five and six, we derive new probability distributions associated with the totally symmetric (bosonic) representations of the C_r, B_r and D_r algebras, respectively. These are the first and simplest results of the second step of the "quantum theory of probability" mentioned above. Since the Dynkin diagram of C_r is obtained from that of A_{2r-1} by folding, the C_r coherent states resemble closely those of the A_{2r-1} algebra. However, the obtained probability distributions, to be denoted as the C_r multinomial distributions, have markedly different features from the ordinary multinomial distributions, reflecting the different weight space structures of the C_r and A_{2r-1} algebras. The probability distributions associated with the symmetric representations of B_r and D_r algebras have also new and interesting features.

Since B_r Dynkin diagram is obtained from that of D_{r+1} by folding, these probability distributions are somewhat related. Section seven is devoted to a summary of results. In the Appendix we give a simple proof and interpretation of another type of addition theorems of Hermite polynomials based on the x representation of $su(1,1)$ and $su(r,1)$ coherent states. The formula is known as generalised Mehler formula but is not found in the standard mathematics reference texts. This time the summation includes infinite number of terms reflecting the infinite dimensionality of the irreducible unitary representations of these non-compact algebras.

2 "Quantum Theory of Probability": An Example

Let us begin with the naive idea of associating "probability amplitude" to a probability distribution. In other words, we explain how to give some meaning to a "square root" of a probability distribution by taking the simplest example of the Poisson distribution. Throughout this paper we consider only discrete probability distributions P parametrised by a set of integers. A probability distribution parametrised by one non-negative integer n is completely specified by a set of non-negative numbers satisfying the conditions of unit total probability:

$$P_n \geq 0, \quad \sum_{n=0}^{\infty} P_n = 1. \tag{2.1}$$

For a quantum theory let us introduce a Hilbert space $\mathcal{H}$ with an orthonormal basis $|n\rangle$, $n = 0, 1, 2, \ldots,$

$$\langle m|n\rangle = \delta_{m\,n}, \tag{2.2}$$

satisfying the completeness relation

$$I = \sum_{n=0}^{\infty} |n\rangle\langle n|, \tag{2.3}$$

in which I on the left hand side is the identity operator. Our objective is to find a normalised state $|\psi\rangle$ in $\mathcal{H}$ such that its transition amplitudes $\langle n|\psi\rangle$ give rise to the probability distribution:

$$|\langle n|\psi\rangle|^2 = P_n, \quad n = 0, 1, 2, \ldots. \tag{2.4}$$

Then by using the completeness relation one obtains

$$|\psi\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\psi\rangle = \sum_{n=0}^{\infty} e^{i\delta_n}\sqrt{P_n}|n\rangle, \tag{2.5}$$

in which the phase δ_n is arbitrary. Thus far the Hilbert space is unspecified.

Let us choose as $\mathcal{H}$ the Hilbert space of one of the simplest quantum systems, the *harmonic oscillator*. It is described by the annihilation and creation operators a and $a^\dagger$ satisfying the commutation relation

$$[a, a^\dagger] = 1. \tag{2.6}$$

(Throughout this paper Planck's constant $\hbar$ is set to unity.) Then the orthonormal basis is simply given by

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle, \quad n = 0, 1, 2, \ldots, \tag{2.7}$$

in which $|0\rangle$ is the vacuum state characterised by the condition

$$a|0\rangle = 0. \tag{2.8}$$

The well-known Poisson distribution describing random processes occurring in a time (space) sequence is

$$P_n(\alpha) = e^{-\alpha^2}\frac{\alpha^{2n}}{n!}, \quad n = 0, 1, 2, \ldots. \tag{2.9}$$

For example, the number of radio-active decay particles emitted from a sample in a fixed time (t) is known to obey this distribution, $\alpha^2 \propto t$. Then the quantum state $|\psi(\alpha)\rangle$ ("probability amplitude") corresponding to the Poisson distribution (2.9) is easily obtained (we set $\delta_n = 0$):

$$|\psi(\alpha)\rangle = e^{-\alpha^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2.10}$$

If we substitute the definition of the number state in terms of the creation operator, we obtain a closed form

$$|\psi(\alpha)\rangle = e^{-\alpha^2/2}e^{\alpha a^\dagger}|0\rangle = e^{\alpha(a^\dagger - a)}|0\rangle, \tag{2.11}$$

and the last formula is obtained by using the Baker-Campbell-Hausdorff (B-C-H) formula

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$$

for the case $[A, B]$ commutes with A and B. This state was first introduced by Schrödinger [1] and discussed by many authors [2, 3, 4] under the name 'coherent state' which was coined by Glauber in quantum optics. The coherent state has many other characterisations.

1. It is an eigenstate of the annihilation operator:

$$a|\psi(\alpha)\rangle = \alpha|\psi(\alpha)\rangle.$$

2. It is a minimum uncertainty state:

$$\langle\Delta x^2\rangle\langle\Delta p^2\rangle = 1/4.$$

in which $x = (a^\dagger + a)/\sqrt{2}$, $p = i(a^\dagger - a)/\sqrt{2}$ are the corresponding coordinate and momentum of the oscillator. Heisenberg's uncertainty principle dictates that

$$\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle \geq 1/4,$$

for arbitrary states.

3. It is obtained by applying a unitary operator (known as the displacement operator)

$$e^{\alpha(a^\dagger - a)}$$

to the vacuum state. Such unitary operators form a (unitary) representation of the Heisenberg-Weyl group.

The last characterisation is generalised by many authors and the concept of the coherent states associated with various Lie algebras (groups) is now well established. Thus starting from a rather naive idea of introducing "probability amplitude" for the Poisson distribution we have arrived at the concept of the coherent states, a rather solid subject in quantum theory and the representation theory of Lie algebras (groups). As we have shown in previous publications [13, 14], the relationship between coherent states and certain probability amplitudes is neither coincidental nor superficial but essential. As we will briefly review in the next section, the "probability amplitudes" for the well-known binomial and multinomial distributions are the coherent states of $su(2)$ and $su(r+1)$ algebras in the totally symmetric (bosonic) representations. The same assertion holds for the negative binomial and negative multinomial distributions and the corresponding algebras are $su(1,1)$ and $su(r,1)$, the non-compact counterparts of $su(2)$ and $su(r+1)$.

3 Coherent States of A_r algebra

3.1 Binomial States

Let us continue along the line of argument of introducing "probability amplitudes" for classical probability distributions. Here we consider the binomial distribution:

$$B_{(n_0,n_1)}(\eta; M) = \binom{M}{n_1} \eta^{2n_1} (1-\eta^2)^{n_0}, \quad n_0 + n_1 = M, \quad \eta \in \mathbf{R}, \tag{3.1}$$

which describes probability distribution of M Bernoulli trials of success (probability η^2) and failure (probability $1-\eta^2$). Here n_1 is the number of successes and n_0 failures. As a Hilbert

space let us choose the Fock space generated by two independent bosonic oscillators:

$$\begin{aligned} [a_j, a_k^\dagger] &= \delta_{jk}, \quad [a_j, a_k] = [a_j^\dagger, a_k^\dagger] = 0, \quad j,k = 0,1, \\ |n_0, n_1\rangle &= \frac{(a_0^\dagger)^{n_0}(a_1^\dagger)^{n_1}}{\sqrt{n_0! n_1!}}|0\rangle, \quad a_j|0\rangle = 0, \quad j = 0,1, \end{aligned} \tag{3.2}$$

and restrict the total number to M (integer)

$$n_0 + n_1 = M. \tag{3.3}$$

Let us denote by $|\eta; M\rangle$ the "square root" of the binomial distribution within this finite $(M+1)$ dimensional Hilbert space. Following the same steps as in the previous section, we arrive at a simple expression:

$$\begin{aligned} |\eta; M\rangle &= \sum_{n_0+n_1=M} |n_0, n_1\rangle\langle n_0, n_1|\eta; M\rangle \\ &= \sum_{n_0+n_1=M} \frac{\sqrt{M!}}{\sqrt{n_0! n_1!}} \eta^{n_1}(1-\eta^2)^{n_0/2}|n_0, n_1\rangle \\ &= \frac{1}{\sqrt{M!}} \sum_{n_0+n_1=M} \frac{M!}{n_0! n_1!}(\eta a_1^\dagger)^{n_1}(\sqrt{1-\eta^2}a_0^\dagger)^{n_0}|0\rangle \\ &= \frac{1}{\sqrt{M!}} \left(\sqrt{1-\eta^2}a_0^\dagger + \eta a_1^\dagger\right)^M |0\rangle, \end{aligned} \tag{3.4}$$

which shows clearly that the "transition amplitude" for each possible result $\langle n_0, n_1|\eta; M\rangle$ is actually obtained by the binomial expansion.

The next step is to identify $|\eta; M\rangle$ as a coherent state. Let us recall the realisation of $su(2)$ algebra in terms of two bosonic oscillators:

$$\begin{aligned} J_+ = a_0^\dagger a_1, \quad J_- &= a_1^\dagger a_0, \quad J_0 = \frac{1}{2}(a_0^\dagger a_0 - a_1^\dagger a_1), \\ [J_+, J_-] &= 2J_0, \quad [J_0, J_\pm] = \pm J_\pm. \end{aligned} \tag{3.5}$$

Obviously the restricted two boson Fock space provides the irreducible (spin $M/2$) representation of $su(2)$ corresponding to the Young diagram

$$\square\square\square\cdots\square\square \quad M \text{ boxes.}$$

Its normalised highest weight state is

$$|M, 0\rangle = \frac{1}{\sqrt{M!}}(a_0^\dagger)^M|0\rangle, \quad J_+|M,0\rangle = 0, \quad J_0|M,0\rangle = \frac{M}{2}|M,0\rangle. \tag{3.6}$$

Similarly to the coherent states of the Heisenberg-Weyl group in the previous section, $su(2)$ coherent states have the form

$$U|\psi_0\rangle, \qquad U \in SU(2). \tag{3.7}$$

These coherent states have "minimal uncertainty" if the 'base' state $|\psi_0\rangle$ corresponds to a dominant weight, i.e., to the highest weight state or its trajectory by the Weyl group [16]. Thus without loss of generality we choose $|\psi_0\rangle = |M,0\rangle$. Since J_+ annihilates the highest weight state and J_0 does not change it, the non-trivial action is by J_- only. So the un-normalised $su(2)$ coherent state is given by

$$e^{\xi J_-}|M,0\rangle = \frac{1}{\sqrt{M!}}e^{\xi a_1^\dagger a_0}(a_0^\dagger)^M|0\rangle = \frac{1}{\sqrt{M!}}(a_0^\dagger+\xi a_1^\dagger)^M|0\rangle, \qquad \xi\in\mathbf{C}. \tag{3.8}$$

Here use is made of the fact that the oscillator algebra $[a_0,a_0^\dagger]=1$ is realised by $a_0=\partial/\partial a_0^\dagger$ and $a_0^\dagger$. At the last equality, the formal Taylor's theorem

$$e^{\alpha\frac{d}{dx}}f(x)=f(x+\alpha) \tag{3.9}$$

is used. It is easy to get the normalised coherent state

$$\frac{1}{M!}\left(\sqrt{1-|\eta|^2}a_0^\dagger+\eta a_1^\dagger\right)^M|0\rangle, \qquad \eta=\xi/\sqrt{1+|\xi|^2}\in\mathbf{C}, \tag{3.10}$$

which has the same form as the binomial state derived above. (In order to get complex η we only have to choose the phase of $\sqrt{B_{(n_0,n_1)}(\eta;M)}$ appropriately.) Thus we have shown that the "probability amplitude" of the binomial distribution is the $su(2)$ coherent state.

3.2 Multinomial States

In this subsection we discuss the relationship between the multinomial distributions and the A_r coherent states [17], which has been demonstrated in some detail in our previous paper [14]. Here we give a simpler and clearer proof of the correspondence with more emphasis on the Lie algebraic structures (i.e., roots and weights) which would be useful for comparison with the results of the other algebras discussed in later sections.

The multinomial distribution is

$$M_{\mathbf{n}}(\boldsymbol{\eta}\,;M)=\frac{M!}{n_0!\cdots n_r!}\,\eta_0^{2n_0}\eta_1^{2n_1}\cdots\eta_r^{2n_r}, \quad n_0+n_1+\cdots+n_r=M, \tag{3.11}$$

in which

$$\mathbf{n}=(n_0,n_1,\ldots,n_r), \quad \eta_0^2=1-\boldsymbol{\eta}^2, \quad 0<\boldsymbol{\eta}^2=\eta_1^2+\cdots+\eta_r^2<1, \quad \eta_j\in\mathbf{R}, \quad j=0,\ldots,r. \tag{3.12}$$

As a Hilbert space let us choose the Fock space generated by $r+1$ independent bosonic oscillators

$$\begin{aligned}[][a_j,a_k^\dagger] &= \delta_{jk}, \qquad a_j|0\rangle=0, \qquad j=0,1,\ldots,r,\\ |\mathbf{n}\rangle &= \frac{(\mathbf{a}^\dagger)^{\mathbf{n}}}{\sqrt{\mathbf{n}!}}|0\rangle, \quad (\mathbf{a}^\dagger)^{\mathbf{n}}=(a_0^\dagger)^{n_0}(a_1^\dagger)^{n_1}\cdots(a_r^\dagger)^{n_r}, \quad \mathbf{n}!=n_0!n_1!\cdots n_r!,\end{aligned} \tag{3.13}$$

and restrict the total number to be M

$$n_0 + n_1 + \cdots + n_r = M. \tag{3.14}$$

It has the dimension

$$\binom{M+r}{M} = \binom{M+r}{r}. \tag{3.15}$$

Let us denote by $|\boldsymbol{\eta}; M\rangle$ the "square root" of the multinomial distribution within this Hilbert space. Then we obtain in a similar way to the binomial state

$$\begin{aligned}
|\boldsymbol{\eta}; M\rangle &= \sum_{n_0+\cdots+n_r=M} |n_0, \cdots, n_r\rangle\langle n_0, \cdots, n_r|\boldsymbol{\eta}; M\rangle \\
&= \sum \frac{\sqrt{M!}}{\sqrt{n_0! \cdots n_r!}}\, \eta_0^{n_0} \cdots \eta_r^{n_r} |n_0, n_1, \cdots, n_r\rangle \\
&= \frac{1}{\sqrt{M!}} \sum \frac{M!}{n_0! n_1! \cdots n_r!}\, (\eta_0 a_0^\dagger)^{n_0} \cdots (\eta_r a_r^\dagger)^{n_r} |0\rangle \\
&= \frac{1}{\sqrt{M!}} \left(\eta_0 a_0^\dagger + \eta_1 a_1^\dagger + \cdots + \eta_r a_r^\dagger\right)^M |0\rangle.
\end{aligned} \tag{3.16}$$

Now let us consider A_r algebra and its representations. Its Dynkin diagram is a simple line connecting r vertices. The number attached to each vertex corresponds to the name of the simple roots given below.

The simple roots are most conveniently expressed in terms of $r+1$ orthonormal vectors in $\mathbf{R}^{r+1}$, $e_j \cdot e_k = \delta_{jk}$, $j, k = 0, 1, \ldots, r$:

$$\alpha_1 = e_0 - e_1, \quad \alpha_2 = e_1 - e_2, \quad \cdots, \quad \alpha_r = e_{r-1} - e_r. \tag{3.17}$$

Then any root, positive or negative, can be expressed as

$$e_j - e_k, \qquad j \neq k, \tag{3.18}$$

which is positive if $j < k$ and negative for $j > k$. All the roots have the same length. The fundamental weight vectors, $\{\lambda_j; j = 1, \ldots, r\}$, the dual basis of the simple root system

$$2\lambda_j \cdot \alpha_k / \alpha_k^2 = \delta_{jk}, \tag{3.19}$$

can also be expressed by $\{e_j\}$. For example

$$\lambda_1 = \frac{1}{r+1}\left(r\,\alpha_1 + (r-1)\alpha_2 + \cdots + \alpha_r\right) = e_0 - \left(e_0 + e_1 + \cdots + e_r\right)/(r+1). \tag{3.20}$$

We consider the irreducible representation of A_r with the highest weight

$$\mu = M\lambda_1 = M\, e_0 - M\,(e_0 + e_1 + \cdots + e_r)\,/(r+1), \tag{3.21}$$

corresponding to the Young diagram

$$\square\square\square\cdots\square\square \quad M \text{ boxes},$$

which has the same dimension $\binom{M+r}{r}$ as the restricted multiboson Fock space introduced above. Thus this completely symmetric representation can be realised in terms of $r+1$ bosonic oscillators. The weights and the occupation numbers are related one to one, namely the state $|n_0, n_1, \ldots, n_r\rangle$ has the weight

$$\mu = \sum_{j=0}^{r} n_j e_j - M\,(e_0 + e_1 + \cdots + e_r)\,/(r+1). \tag{3.22}$$

All the weight spaces are non-degenerate, i.e., one-dimensional.

If we denote the A_r generators corresponding to the root $e_j - e_k$ by $X_{(j,-k)}$, we have

$$X_{(j,-k)} = a_j^\dagger a_k \tag{3.23}$$

and

$$\begin{aligned} [X_{(j,-k)}, X_{(k,-l)}] &= [a_j^\dagger a_k, a_k^\dagger a_l] = a_j^\dagger a_l = X_{(j,-l)}, \\ [X_{(j,-k)}, X_{(k,-j)}] &= H_{(j,k)} \equiv a_j^\dagger a_j - a_k^\dagger a_k. \end{aligned} \tag{3.24}$$

Here $H_{(j,k)}$ belongs to the Cartan subalgebra. The quadratic Casimir operator is

$$\mathbf{C}_2 = \frac{r}{r+1} N_{tot}(N_{tot} + r + 1), \qquad N_{tot} = \sum_{j=0}^{r} a_j^\dagger a_j, \tag{3.25}$$

which takes the value $rM(M+r+1)/(r+1)$ in the present representation. The state having the highest weight (3.21) is

$$|M, 0, \ldots, 0\rangle = \frac{(a_0^\dagger)^M}{\sqrt{M!}}|0\rangle, \tag{3.26}$$

which is annihilated by the generators

$$X_{(j,k)}, \qquad H_{(j,k)}, \qquad j,k = 1, \ldots, r, \tag{3.27}$$

forming an A_{r-1} subalgebra. The action of the Cartan subalgebra generators $H_{(0,j)}$ does not change the state, either:

$$H_{(0,j)}|M, 0, \ldots, 0\rangle = M|M, 0, \ldots, 0\rangle.$$

Thus the coherent states based on the highest weight state (3.21) are characterised by

$$SU(r+1)/U(1) \times SU(r) = \mathbf{CP}^r. \tag{3.28}$$

Among the generators belonging to $\mathbf{CP}^r$, only those

$$X_{(j,-0)} = a_j^\dagger a_0, \qquad j = 1, \ldots, r \tag{3.29}$$

have non-trivial action on the highest weight state (3.21). Thus we find, as in the case of the binomial state (3.8), that the un-normalised A_r coherent state is expressed as

$$\begin{aligned} & e^{\sum_{j=1}^r \xi_j X_{(j,-0)}} |M, 0, \ldots, 0\rangle \\ = & \frac{1}{\sqrt{M!}} e^{(\sum_{j=1}^r \xi_j a_j^\dagger) a_0} (a_0^\dagger)^M |0\rangle \\ = & \frac{1}{\sqrt{M!}} \left(a_0^\dagger + \sum_{j=1}^r \xi_j a_j^\dagger \right)^M |0\rangle, \qquad \boldsymbol{\xi} = (\xi_1, \ldots, \xi_r) \in \mathbf{CP}^r, \end{aligned} \tag{3.30}$$

in which use has been made of the Taylor expansion theorem (3.9) with $a_0 = \partial/\partial a_0^\dagger$.

The normalised A_r coherent state in the totally symmetric representation is given by

$$|\boldsymbol{\eta}; M\rangle = \frac{1}{\sqrt{M!}} \left(\eta_0 a_0^\dagger + \sum_{j=1}^r \eta_j a_j^\dagger \right)^M |0\rangle, \qquad \eta_j = \xi_j / \sqrt{1 + |\boldsymbol{\xi}|^2} \in \mathbf{C}, \quad \eta_0 = \sqrt{1 - |\boldsymbol{\eta}|^2}, \tag{3.31}$$

which has the same form as the multinomial state $|\boldsymbol{\eta}; M\rangle$ derived above. As in the binomial state case the "transition amplitude" $\langle n_0, \ldots, n_r | \boldsymbol{\eta}; M\rangle$ to each number state (or weight state $\langle \mu_1, \ldots, \mu_r | \boldsymbol{\eta}; M\rangle$) is simply obtained by multinomial expansion.

3.3 Coordinate Representation and Addition Theorems of Hermite Polynomials I

In this subsection we consider the 'coordinate representation' of the multinomial state (3.31). This representation is useful in quantum optics. It also gives a simple proof and interpretation of the following addition theorem of Hermite polynomials (see, for example, [18] and p196 of [19]):

$$\begin{aligned} & \frac{(\eta_0^2 + \cdots + \eta_r^2)^{M/2}}{M!} H_M \left((\eta_0 x_0 + \cdots + \eta_r x_r) / \sqrt{\eta_0^2 + \cdots + \eta_r^2} \right) \\ = & \sum_{n_0 + \cdots + n_r = M} \frac{\eta_0^{n_0}}{n_0!} \cdots \frac{\eta_r^{n_r}}{n_r!} H_{n_0}(x_0) \cdots H_{n_r}(x_r). \end{aligned} \tag{3.32}$$

Here $\eta_0,\ldots,\eta_r$ are arbitrary complex numbers. It should be noted that the left hand side contains $\sqrt{\eta_0^2+\cdots+\eta_r^2}$ in even powers only, since Hermite polynomials have a definite parity:

$$H_M(-x)=(-1)^M H_M(x).$$

Let us begin with a single boson oscillator

$$[a,a^\dagger]=1.$$

The coordinate representation of the number state $|n\rangle$ is

$$\langle x|n\rangle=\frac{1}{\sqrt{n!}}\langle x|(a^\dagger)^n|0\rangle=\frac{1}{\pi^{1/4}2^{n/2}\sqrt{n!}}H_n(x)\,e^{-\frac{1}{2}x^2}, \tag{3.33}$$

in which Hermite polynomial H_n is given by Rodrigues formula:

$$H_n(x)=(-1)^n e^{x^2}D^n e^{-x^2},\quad D=\frac{d}{dx}. \tag{3.34}$$

It is well-known that the generating function of the Hermite polynomials

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x)=e^{-t^2+2tx} \tag{3.35}$$

is essentially the same as the coordinate representation of the coherent state of the Heisenberg-Weyl group (2.10):

$$\langle x|\psi(\alpha)\rangle=e^{-\frac{1}{2}(x-\sqrt{2}\alpha)^2}/\pi^{1/4},\qquad \alpha\in\mathbf{R}. \tag{3.36}$$

The coordinate representation of the multinomial state (3.31) is simply obtained by expansion ($\eta_1,\ \ldots,\eta_r$ are in general complex):

$$\begin{aligned}
&\langle x_0,x_1,\ldots,x_r|\boldsymbol{\eta};M\rangle\\
=&\ \frac{1}{\sqrt{M!}}\langle x_0,x_1,\ldots,x_r|\left(\eta_0a_0^\dagger+\cdots+\eta_ra_r^\dagger\right)^M|0\rangle\\
=&\ \sqrt{M!}\,\frac{e^{-\frac{1}{2}(x_0^2+\cdots+x_r^2)}}{\pi^{(r+1)/4}2^{M/2}}\sum_{n_0+\cdots+n_r=M}\frac{\eta_0^{n_0}}{n_0!}\cdots\frac{\eta_r^{n_r}}{n_r!}H_{n_0}(x_0)\cdots H_{n_r}(x_r).
\end{aligned} \tag{3.37}$$

Next we consider operators A and $\widetilde{A}$ defined by

$$A=\frac{\eta_0a_0+\cdots+\eta_ra_r}{\sqrt{\eta_0^2+\cdots+\eta_r^2}},\qquad \widetilde{A}=\frac{\eta_0a_0^\dagger+\cdots+\eta_ra_r^\dagger}{\sqrt{\eta_0^2+\cdots+\eta_r^2}}. \tag{3.38}$$

They are not hermitian conjugate of each other but they satisfy the same relations as those of the single oscillator:

$$[A,\widetilde{A}]=1,\qquad A|0\rangle=0,$$

which are essential for deriving Hermite polynomials. Thus we obtain

$$\begin{aligned}
&\langle x_0, x_1, \ldots, x_r|\boldsymbol{\eta}; M\rangle \\
= \;& \frac{(\eta_0^2+\cdots+\eta_r^2)^{M/2}}{\sqrt{M!}}\langle x_0, x_1, \ldots, x_r|\tilde{A}^M|0\rangle \\
= \;& \frac{(\eta_0^2+\cdots+\eta_r^2)^{M/2}}{\sqrt{M!}}\frac{e^{-\frac{1}{2}(x_0^2+\cdots+x_r^2)}}{\pi^{(r+1)/4}2^{M/2}}H_M\left((\eta_0x_0+\cdots+\eta_rx_r)/\sqrt{\eta_0^2+\cdots+\eta_r^2}\right). \quad (3.39)
\end{aligned}$$

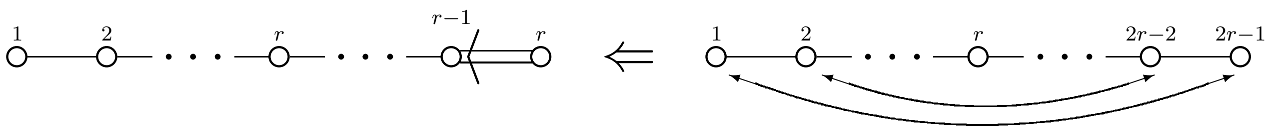
Comparing (3.37) and (3.39) we obtain the above mentioned addition theorem (3.32) of Hermite polynomials, which is nothing but the multinomial expansion of the multinomial state. In the Appendix we give a proof and interpretation of another type of addition theorems of Hermite polynomials based on negative multinomial states, i.e., the coherent states of $su(r,1)$ algebra in discrete symmetric representations.

4 C_r Multinomial States

Let us proceed to the second step in the study of "quantum probability". In the previous sections we have shown that some of the typical discrete probability distributions are characterised by Lie algebras through coherent states. Now we reverse the logic and try to derive new probability distributions starting from Lie algebras and their representations. For this we have, in principle, an infinite choice of Lie algebras and their representations. Probably most of such new probability distributions are too exotic to have any practical use at the moment. However, the great role played by the Poisson, the binomial, the multinomial distributions and their "negative" (non-compact) counterparts makes us expect that the probability distributions related with the totally symmetric representations of the other classical algebras, B_r, C_r and D_r could be useful, though possibly to a lesser degree. Apart from the Poisson distribution which has only one parameter, the (negative) multinomial distribution has many parameters, $\boldsymbol{\eta}$ and M, to give suitable description to various statistical phenomena. The same property is shared by all the probability distributions derived from the totally symmetric representations of B_r, C_r and D_r algebras. We propose to call these coherent states the B_r, C_r and D_r *multinomial states* and the corresponding probability distributions the B_r, C_r and D_r *multinomial distributions*. We start with the C_r case and proceed to D_r and B_r cases, in the order of increasing complexity.

4.1 Coherent States

The Dynkin diagram of C_r is obtained from that of A_{2r-1} by folding.



Its simple roots can be expressed most conveniently in terms of an orthonormal basis of $\mathbf{R}^r$, $e_j \cdot e_k = \delta_{jk}$, $j, k = 0, \ldots, r$:

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \ \cdots, \ \alpha_{r-1} = e_{r-1} - e_r, \quad \alpha_r = 2e_r. \tag{4.1}$$

The positive roots are

$$e_j - e_k, \qquad (j < k), \qquad e_j + e_k, \qquad 2e_j. \tag{4.2}$$

There are $2r(r-1)$ short roots and $2r$ long roots ($\pm 2e_j$) and the dimensions of C_r algebra is $2r^2 + r$. The fundamental weights are

$$\lambda_1 = e_1, \qquad \lambda_2 = e_1 + e_2, \qquad \ldots \tag{4.3}$$

We consider the irreducible representation with the highest weight

$$\mu = M\lambda_1 = Me_1. \tag{4.4}$$

Its dimensionality is

$$\binom{M + 2r - 1}{2r - 1} = \binom{M + 2r - 1}{M}.$$

It is the same as the dimension of the restricted multiboson (M particle) Fock space of A_{2r-1} with $2r$ bosonic oscillators:

$$[a_j, a_k^\dagger] = [b_j, b_k^\dagger] = \delta_{jk}, \qquad j, k = 1, \ldots, r \tag{4.5}$$

with the number states

$$|n_1, \ldots, n_r; \bar{n}_1, \ldots, \bar{n}_r\rangle, \qquad n_1 + \cdots + n_r + \bar{n}_1 + \cdots + \bar{n}_r = M, \tag{4.6}$$

in which n_j ($\bar{n}_j$) is the number of a_j (b_j) quanta.

Similarly to the A_r case, we introduce the following notation for the generators corresponding to the roots:

$$\begin{aligned} X_{(j,-k)} \ &\Leftrightarrow \ e_j - e_k, \\ X_{(j,k)} \ &\Leftrightarrow \ e_j + e_k, \quad X_{(-j,-k)} \Leftrightarrow -e_j - e_k, \\ X_{(j,j)} \ &\Leftrightarrow \ 2e_j, \qquad\quad X_{(-j,-j)} \Leftrightarrow -2e_j. \end{aligned} \tag{4.7}$$

Their forms are

$$
\begin{aligned}
X_{(j,-k)} &= a_j^\dagger a_k - b_k^\dagger b_j, \\
X_{(j,k)} &= a_j^\dagger b_k + a_k^\dagger b_j, \quad X_{(-j,-k)} = b_j^\dagger a_k + b_k^\dagger a_j, \\
X_{(j,j)} &= a_j^\dagger b_j, \qquad\qquad\quad X_{(-j,-j)} = b_j^\dagger a_j.
\end{aligned} \tag{4.8}
$$

It is elementary to check the commutation relations, for example:

$$
\begin{aligned}
[X_{(j,-k)}, X_{(k,-l)}] &= [a_j^\dagger a_k - b_k^\dagger b_j, a_k^\dagger a_l - b_l^\dagger b_k] = a_j^\dagger a_l - b_l^\dagger b_j = X_{(j,-l)}, \\
[X_{(j,-k)}, X_{(k,-j)}] &= a_j^\dagger a_j - b_j^\dagger b_j - a_k^\dagger a_k + b_k^\dagger b_k \equiv H_j - H_k, \quad etc.
\end{aligned} \tag{4.9}
$$

The quadratic Casimir operator is

$$
C_2 = N_{tot}(N_{tot} + 2r), \qquad N_{tot} = \sum_{j=1}^{r} (a_j^\dagger a_j + b_j^\dagger b_j), \tag{4.10}
$$

which gives $M(M+2r)$ in the present representation. It is easy to see that each number state belongs to some weight

$$
|n_1, \ldots, n_r; \bar{n}_1, \ldots, \bar{n}_r\rangle \Rightarrow \mu = \sum_{j=1}^{r} (n_j - \bar{n}_j) e_j. \tag{4.11}
$$

In contradistinction with the A_{2r-1} case this correspondence is not 1 to 1. Some weight spaces are degenerate. For example for $M = 4$ and $r = 2$,

$$
|1,1;1,1\rangle, \quad |2,0;2,0\rangle, \quad |0,2;0,2\rangle
$$

belong to the null weight $\mu = 0$.

As in the case of the binomial states (3.7) we adopt as the 'base' state $|\psi_0\rangle$ the highest weight state

$$
|M, 0, \ldots, 0; 0, \ldots, 0\rangle = \frac{(a_1^\dagger)^M}{\sqrt{M!}} |0\rangle, \tag{4.12}
$$

which guarantees "minimum uncertainty". Together with all the positive root generators, it is also annihilated by the following generators:

$$
X_{(j,-k)}, \quad X_{(j,k)}, \quad X_{(-j,-k)}, \quad X_{(j,j)}, \quad X_{(-j,-j)}, \quad H_j, \qquad 2 \le j, k \le r, \tag{4.13}
$$

which form a C_{r-1} subalgebra. Likewise the action of the Cartan subalgebra generator H_1 does not change the highest weight state. Therefore the C_r multinomial states are parametrised by

$$
Sp(2r)/U(1) \times Sp(2(r-1)) = \mathbf{CP}^{2r-1},
$$

which also indicates the connection to the A_{2r-1} case. In fact the generators having non-trivial action on the highest weight state are

$$X_{(-1,j)}, \quad 2 \leq j \leq r \quad \text{and} \quad X_{(-1,-j)}, \quad 1 \leq j \leq r. \tag{4.14}$$

The generators in the first (second) group commute among themselves. In particular, $X_{(-1,-1)}$ which belongs to the lowest root, commutes with all the generators in the list (4.14). The non-commuting pairs among the above generators are

$$[X_{(-1,j)}, X_{(-1,-j)}] = -2X_{(-1,-1)}, \qquad 2 \leq j \leq r, \tag{4.15}$$

and the resulting generator commutes with all the other generators in the list (4.14), as shown above.

In terms of $2r-1$ complex parameters

$$\xi_j,\ 2 \leq j \leq r, \quad \xi_{-j},\ 1 \leq j \leq r, \quad \boldsymbol{\xi} = (\xi_2, \ldots, \xi_r; \xi_{-1}, \ldots, \xi_{-r}) \in \mathbf{CP}^{2r-1}, \tag{4.16}$$

the un-normalised coherent state is expressed as

$$e^{C+D}(a_1^\dagger)^M|0\rangle, \quad C = \sum_{j=2}^{r} \xi_j X_{(-1,j)}, \quad D = \sum_{j=1}^{r} \xi_{-j} X_{(-1,-j)}, \tag{4.17}$$

with $[C, D] = 2(\sum_{j=2}^{r} \xi_j \xi_{-j}) X_{(-1,-1)}$ commuting with C and D. With the help of the B-C-H formula

$$e^{C+D} = e^{C - \frac{1}{2}[C,D]} e^{D}$$

and the formal Taylor expansion theorem (3.9) we arrive at the following expression of the un-normalised C_r multinomial state

$$\left(a_1^\dagger + \sum_{j=2}^{r} \xi_j a_j^\dagger + \sum_{j=1}^{r} \xi_{-j} b_j^\dagger \right)^M |0\rangle, \tag{4.18}$$

in which the effects of non-commutativity cancel out exactly. Therefore the normalised C_r multinomial state is

$$|\boldsymbol{\eta}; M; C_r\rangle = \frac{1}{\sqrt{M!}} \left(\sum_{j=1}^{r} \eta_j a_j^\dagger + \sum_{j=1}^{r} \eta_{-j} b_j^\dagger \right)^M |0\rangle, \tag{4.19}$$

in which

$$\eta_1 = \left(1 + \sum_{j=2}^{r} |\xi_j|^2 + \sum_{j=1}^{r} |\xi_{-j}|^2 \right)^{-\frac{1}{2}}, \quad \eta_j = \xi_j \eta_1, \quad \eta_{-j} = \xi_{-j} \eta_1, \qquad 2 \leq j \leq r, \tag{4.20}$$

satisfying the condition

$$\sum_{j=1}^{r} (|\eta_j|^2 + |\eta_{-j}|^2) = 1.$$

This has exactly the same form as the A_{2r-1} multinomial state.

4.2 Probability Distribution

Now we derive the probability distribution from the coherent state, which has exactly the same form as the A_r multinomial state. So it predicts the multinomial distribution for the *numbers* $n_1,\ldots,\bar{n}_r$ with the corresponding probabilities $|\eta_1|^2,\ldots,|\eta_{-r}|^2$:

$$|\langle n_1,\ldots,n_r;\bar{n}_1,\ldots,\bar{n}_r|\boldsymbol{\eta};M;C_r\rangle|^2 = \frac{M!}{n_1!\cdots n_r!\bar{n}_1!\cdots\bar{n}_r!}|\eta_1|^{2n_1}\cdots|\eta_r|^{2n_r}|\eta_{-1}|^{2\bar{n}_1}\cdots|\eta_{-r}|^{2\bar{n}_r}. \tag{4.21}$$

As remarked above, the C_r states are labeled by the weight

$$\boldsymbol{\mu} = (\mu_1,\ldots,\mu_r)$$

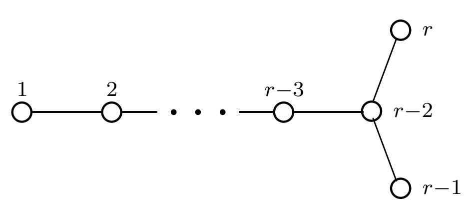
which takes positive, zero and negative integer values. Each weight space has one or many number states which are orthogonal to each other. Therefore the C_r multinomial distribution is obtained by summing the contributions from these number states:

$$C_{\boldsymbol{\mu}}(\boldsymbol{\eta};M) = \sum_{n_j-\bar{n}_j=\mu_j}\frac{M!}{n_1!\cdots n_r!\bar{n}_1!\cdots\bar{n}_r!}|\eta_1|^{2n_1}\cdots|\eta_r|^{2n_r}|\eta_{-1}|^{2\bar{n}_1}\cdots|\eta_{-r}|^{2\bar{n}_r}. \tag{4.22}$$

Let us interpret it in terms of "picking up balls from a pot". The pot contains an infinite number of balls of r-different colours. There are two types of balls for each colour, the "*positive*" one and "*negative*" one. Let the probabilities of picking one j-th colour ball be η_j^2 for the "*positive*" and η_{-j}^2 for the "*negative*". We pick up total of M balls and ask the probability distribution for the "net" number of balls (or the "weight") for each colour: $\mu_j = n_j - \bar{n}_j$, $j = 1,\ldots,r$. It is given by the C_r multinomial distribution. We see that the folding of the A_{2r-1} Dynkin diagram leading to that of C_r is very suggestive of this situation.

5 D_r Multinomial States

Here we will derive probability distributions associated with the symmetric representations of D_r algebra. They have some new features not present in the multinomial distributions associated with A_{2r-1} or C_r algebras. The Dynkin diagram of D_r algebra with the names of simple roots attached to the vertices is shown below.



The corresponding simple roots are

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ldots,\alpha_{r-2} = e_{r-2} - e_{r-1},\ \alpha_{r-1} = e_{r-1} - e_r,\ \alpha_r = e_{r-1} + e_r. \tag{5.1}$$

The positive roots are all of the same length:

$$e_j - e_k \quad (j < k), \qquad e_j + e_k. \tag{5.2}$$

The dimension of D_r algebra is $2r^2 - r$. The fundamental weights are

$$\lambda_1 = e_1, \quad \lambda_2 = e_1 + e_2, \ldots, \tag{5.3}$$

and we consider, as before, the irreducible representation with highest weight

$$\mu = M\lambda_1 = Me_1. \tag{5.4}$$

Let us denote this representation by ρ_D^M and the corresponding vector space by V_D^M. We know from Weyl's dimension formula

$$\dim(V_D^M) = \binom{M+2r-3}{2r-3} \times \frac{M+r-1}{r-1}. \tag{5.5}$$

Let us realise this representation in terms of $2r$ bosons

$$a_1, \ldots, a_r, \quad b_1, \ldots, b_r,$$

and in its restricted Fock space denoted by F_{2r}^M,

$$F_{2r}^M; \quad |n_1, \ldots, n_r; \bar{n}_1, \ldots, \bar{n}_r\rangle, \qquad n_1 + \cdots + n_r + \bar{n}_1 + \cdots + \bar{n}_r = M. \tag{5.6}$$

We have

$$\dim(F_{2r}^M) = \binom{M+2r-1}{2r-1} = \binom{M+2r-1}{M}. \tag{5.7}$$

Comparing (5.5) and (5.7), we find

$$\begin{aligned} \dim(F_{2r}^M) &= \dim(V_D^M) + \dim(F_{2r}^{M-2}) \\ &= \dim(V_D^M) + \dim(V_D^{M-2}) + \cdots, \end{aligned} \tag{5.8}$$

which means that the bosonic Fock space F_{2r}^M contains several irreducible representations ρ_D^L with different L's.

Let us introduce, as in the C_r case, the following notation for the generators corresponding to the roots:

$$\begin{aligned} X_{(j,-k)} &\Leftrightarrow e_j - e_k, \\ X_{(j,k)} &\Leftrightarrow e_j + e_k, \quad X_{(-j,-k)} \Leftrightarrow -e_j - e_k. \end{aligned} \tag{5.9}$$

Their forms are

$$
\begin{aligned}
X_{(j,-k)} &= a_j^\dagger a_k - b_k^\dagger b_j, \\
X_{(j,k)} &= a_j^\dagger b_k - a_k^\dagger b_j, \quad X_{(-j,-k)} = b_k^\dagger a_j - b_j^\dagger a_k.
\end{aligned} \tag{5.10}
$$

It is elementary to check the commutation relations, for example they are (4.9) and:

$$
\begin{aligned}
[X_{(j,-k)}, X_{(k,l)}] &= [a_j^\dagger a_k - b_k^\dagger b_j, a_k^\dagger b_l - a_l^\dagger b_k] = a_j^\dagger b_l - a_l^\dagger b_j = X_{(j,l)}, \\
[X_{(j,k)}, X_{(-j,-k)}] &= a_j^\dagger a_j - b_j^\dagger b_j + a_k^\dagger a_k - b_k^\dagger b_k \equiv H_j + H_k, \quad \textit{etc.}
\end{aligned} \tag{5.11}
$$

The quadratic Casimir operator is

$$
C_2 = N_{tot}\left(N_{tot} + 2(r-1)\right) - 4K^\dagger K, \qquad N_{tot} = \sum_{j=1}^{r}(a_j^\dagger a_j + b_j^\dagger b_j), \tag{5.12}
$$

in which K and $K^\dagger$ are quadratic operators in the oscillators

$$
K = \sum_{j=1}^{r} a_j b_j, \qquad K^\dagger = \sum_{j=1}^{M} a_j^\dagger b_j^\dagger. \tag{5.13}
$$

They commute with all the above generators including those belonging to the Cartan subalgebra:

$$
[K, X_{\pm(j,\pm k)}] = [K, H_j] = [K^\dagger, X_{\pm(j,\pm k)}] = [K^\dagger, H_j] = 0. \tag{5.14}
$$

In terms of $K^\dagger$ we can express the decomposition of the bosonic Fock space succinctly:

$$
F_{2r}^M = V_D^M \oplus V_D^{M-2} \oplus \cdots V_D^1(V_D^0), \tag{5.15}
$$

in which the vector space V_D^M is obtained from the highest weight state

$$
|M, 0, \ldots, 0; 0, \ldots, 0\rangle = \frac{(a_1^\dagger)^M}{\sqrt{M!}}|0\rangle, \tag{5.16}
$$

by applying the negative weight generators successively. The j-th vector space in the right hand side $V_D^{M-2(j-1)}$ is obtained from the highest weight state

$$
\frac{(a_1^\dagger)^{M-2(j-1)}}{\sqrt{(M-2(j-1))!}}\,(K^\dagger)^{j-1}|0\rangle, \tag{5.17}
$$

by applying the negative weight generators successively. It is easy to see that K annihilates all the states in V_D^M

$$
Kv = 0, \quad \forall v \in V_D^M,
$$

and we get $C_2 = M(M + 2(r-1))$ in the highest weight representation (5.4),(5.16). It is easy to see that each number state belongs to some weight

$$
|n_1, \ldots, n_r; \bar{n}_1, \ldots, \bar{n}_r\rangle \Rightarrow \mu = \sum_{j=1}^{r}(n_j - \bar{n}_j)e_j. \tag{5.18}
$$

The highest weight state (5.17) is annihilated by the following generators belonging to a D_{r-1} subalgebra

$$X_{(j,-k)}, \quad X_{(j,k)}, \quad X_{(-j,-k)}, \quad H_j, \qquad 2 \le j,k \le r, \tag{5.19}$$

as well as by all the positive root generators. The Cartan subalgebra generator H_1 does not change the highest weight state. In other words, the generators having non-trivial action on the highest weight state are

$$X_{(-1,j)}, \quad X_{(-1,-j)}, \quad 2 \le j \le r. \tag{5.20}$$

If we denote the compact group corresponding to D_r by $SO(2r)$, the D_r multinomial states are parametrised by

$$SO(2r)/U(1) \times SO(2(r-1)),$$

having the dimension

$$4(r-1).$$

In terms of $2(r-1)$ complex parameters

$$\xi_j, \quad \xi_{-j}, \quad 2 \le j \le r, \tag{5.21}$$

we define a linear combination of the non-trivial generators (5.20) as

$$T = \sum_{j=2}^{r} \xi_j X_{(-1,j)} + \sum_{j=2}^{r} \xi_{-j} X_{(-1,-j)}. \tag{5.22}$$

It should be noted that all the generators in (5.22) or (5.20) commute among themselves, since the sum of the corresponding roots are not roots any more. Thus we arrive at the expression of the un-normalised coherent state:

$$\exp[T](a_1^\dagger)^M|0\rangle = \prod_{j=2}^{r} \exp(\xi_j X_{(-1,j)}) \prod_{j=2}^{r} \exp(\xi_{-j} X_{(-1,-j)})(a_1^\dagger)^M|0\rangle. \tag{5.23}$$

By repeated use of the formal Taylor expansion theorem (3.9) we obtain the following explicit form

$$\left(a_1^\dagger + \sum_{j=2}^{r} \xi_j a_j^\dagger + \sum_{j=2}^{r} \xi_{-j} b_j^\dagger - (\sum_{j=2}^{r} \xi_j \xi_{-j}) b_1^\dagger\right)^M |0\rangle. \tag{5.24}$$

This looks similar to the A_{2r-1} and C_r multinomial states, except that the coefficient of $b_1^\dagger$ is not independent. The normalised D_r multinomial state is

$$|\boldsymbol{\eta}; M; D_r\rangle = \frac{1}{\sqrt{M!}} \left(\sum_{j=1}^{r} \eta_j a_j^\dagger + \sum_{j=1}^{r} \eta_{-j} b_j^\dagger\right)^M |0\rangle, \tag{5.25}$$

in which

$$\eta_1 = \left(1+\sum_{j=2}^{r}|\xi_j|^2+\sum_{j=2}^{r}|\xi_{-j}|^2+|\sum_{j=2}^{r}\xi_j\xi_{-j}|^2\right)^{-\frac{1}{2}}, \ \eta_j=\xi_j\eta_1, \quad \eta_{-j}=\xi_{-j}\eta_1, \ 2\le j\le r,$$
$$\eta_{-1} = -(\sum_{j=2}^{r}\xi_j\xi_{-j})\eta_1, \tag{5.26}$$

satisfying the condition

$$\sum_{j=1}^{r}(|\eta_j|^2+|\eta_{-j}|^2)=1.$$

Let us turn to the form of the probability distribution derived from the D_r multinomial state, which has a form similar to that derived from the A_r multinomial state. Similar to the C_r case, D_r multinomial state predicts the multinomial distribution to the *number states* with the probabilities $|\eta_j|^2$ and $|\eta_{-j}|^2$:

$$|\langle n_1,\ldots,n_r;\bar{n}_1,\ldots,\bar{n}_r|\boldsymbol{\eta};M;D_r\rangle|^2=\frac{M!}{n_1!\cdots n_r!\bar{n}_1!\cdots\bar{n}_r!}|\eta_1|^{2n_1}\cdots|\eta_r|^{2n_r}|\eta_{-1}|^{2\bar{n}_1}\cdots|\eta_{-r}|^{2\bar{n}_r}. \tag{5.27}$$

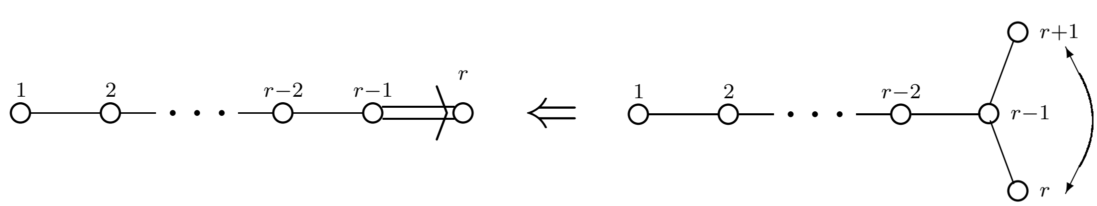
By summing the contributions from all the number states belonging to a given weight $\boldsymbol{\mu}$ we obtain D_r multinomial distribution:

$$D_{\boldsymbol{\mu}}(\boldsymbol{\eta};M)=\sum_{n_j-\bar{n}_j=\mu_j}\frac{M!}{n_1!\cdots n_r!\bar{n}_1!\cdots\bar{n}_r!}|\eta_1|^{2n_1}\cdots|\eta_r|^{2n_r}|\eta_{-1}|^{2\bar{n}_1}\cdots|\eta_{-r}|^{2\bar{n}_r}. \tag{5.28}$$

Thus the interpretation as "picking up coloured balls from a pot" is also valid. The marked difference is that among the probabilities $|\eta_1|^2, \ldots, |\eta_r|^2, |\eta_{-1}|^2, \ldots, |\eta_{-r}|^2$, only $2(r-1)$ of them are independent. As is clear from (5.26), one of the *dependent* probabilities, say $|\eta_{-1}|^2$, depends on the information of the other $\eta_{\pm j}$'s including their phases (or more precisely ξ_j's) , not $|\eta_{\pm j}|^2$'s. We believe that this is a novel feature not encountered in any classical probability distributions. We may say that the D_r multinomial distribution has non-classical (or quantum) features.

6 B_r Multinomial States

The Dynkin diagram of B_r is obtained from that of D_{r+1} by folding the two tails.



Thus we expect that the B_r multinomial states (distributions) have similarities with those of D_r with some added new features due to the folding. The simple roots of B_r are

$$\alpha_1 = e_1 - e_2, \quad \alpha_2 = e_2 - e_3, \ \cdots, \ \alpha_{r-1} = e_{r-1} - e_r, \quad \alpha_r = e_r. \tag{6.1}$$

The positive roots are

$$e_j - e_k, \qquad (j < k), \qquad e_j + e_k, \qquad e_j. \tag{6.2}$$

There are $2r(r-1)$ long roots and $2r$ short roots $(\pm e_j)$ and the dimension of B_r algebra is $2r^2 + r$, the same as C_r. The fundamental weights are

$$\lambda_1 = e_1, \qquad \lambda_2 = e_1 + e_2, \qquad \ldots \tag{6.3}$$

As before we consider the irreducible representation with the highest weight

$$\mu = M\lambda_1 = Me_1. \tag{6.4}$$

Let us denote this representation ρ_B^M and the corresponding vector space by V_B^M. Weyl's dimension formula gives

$$\dim(V_B^M) = \binom{M + 2r - 2}{2r - 2} \times \frac{2M + 2r - 1}{2r - 1}. \tag{6.5}$$

This representation is realised in a restricted Fock space denoted by F_{2r+1}^M:

$$F_{2r+1}^M; \quad |n_0, n_1, \ldots, n_r; \bar{n}_1, \ldots, \bar{n}_r\rangle, \qquad n_0 + n_1 + \cdots + n_r + \bar{n}_1 + \cdots + \bar{n}_r = M, \tag{6.6}$$

which is generated by $2r + 1$ bosonic oscillators

$$a_0, a_1, \ldots, a_r, \quad b_1, \ldots, b_r.$$

As in the D_r case, by comparing the dimensions of the bosonic Fock space

$$\dim(F_{2r+1}^M) = \binom{M + 2r}{2r} = \binom{M + 2r}{M} \tag{6.7}$$

with the dimensions of V_B^M (6.5), we find

$$\begin{aligned} \dim(F_{2r+1}^M) &= \dim(V_B^M) + \dim(F_{2r+1}^{M-2}) \\ &= \dim(V_B^M) + \dim(V_B^{M-2}) + \cdots, \end{aligned} \tag{6.8}$$

which means that the bosonic Fock space F_{2r+1}^M contains several irreducible representations ρ_B^L with different highest weights $(L = M, M-2, \ldots,)$.

Similarly to the A_r case, the generators corresponding to various roots have the following forms:

$$\begin{aligned} X_{(j,-k)} &= a_j^\dagger a_k - b_k^\dagger b_j, \\ X_{(j,k)} &= a_j^\dagger b_k - a_k^\dagger b_j, \quad X_{(-j,-k)} = b_j^\dagger a_k - b_k^\dagger a_j, \\ X_{(j,0)} &= a_j^\dagger a_0 - a_0^\dagger b_j, \quad X_{(-j,-j)} = a_0^\dagger a_j - b_j^\dagger a_0, \end{aligned} \tag{6.9}$$

in which, as in the C_r case, we use the notation:

$$\begin{aligned} X_{(j,-k)} &\Leftrightarrow e_j - e_k, \\ X_{(j,k)} &\Leftrightarrow e_j + e_k, \quad X_{(-j,-k)} \Leftrightarrow -e_j - e_k, \\ X_{(j,0)} &\Leftrightarrow e_j, \qquad\quad X_{(-j,0)} \Leftrightarrow -e_j. \end{aligned} \tag{6.10}$$

The commutation relations are easily verified as in the previous cases. The quadratic Casimir operator is

$$C_2 = N_{tot}(N_{tot} + 2r - 1) - 4K^\dagger K, \qquad N_{tot} = a_0^\dagger a_0 + \sum_{j=1}^{r}(a_j^\dagger a_j + b_j^\dagger b_j), \tag{6.11}$$

in which K and $K^\dagger$ are quadratic operators in the oscillators

$$K = \frac{1}{2}a_0^2 + \sum_{j=1}^{r} a_j b_j, \qquad K^\dagger = \frac{1}{2}(a_0^\dagger)^2 + \sum_{j=1}^{M} a_j^\dagger b_j^\dagger. \tag{6.12}$$

As in the D_r cases, K and $K^\dagger$ commute with all the above generators including those belonging to the Cartan subalgebra. The decomposition of the restricted bosonic Fock space into the irreducible representation spaces goes in parallel with the D_r case:

$$F_{2r+1}^M = V_B^M \oplus V_B^{M-2} \oplus \cdots V_B^1(V_B^0), \tag{6.13}$$

in which the vector space V_B^M is obtained from the highest weight state

$$\frac{1}{\sqrt{M!}}(a_1^\dagger)^M|0\rangle = |0, M, 0, \ldots; 0, \ldots, 0\rangle, \tag{6.14}$$

by applying the negative root generators successively. The j-th vector space in the right hand side $V_B^{M-2(j-1)}$ is obtained from the highest weight state

$$\frac{(a_1^\dagger)^{M-2(j-1)}}{\sqrt{(M-2(j-1))!}}(K^\dagger)^{j-1}|0\rangle, \tag{6.15}$$

in a similar way. As in the D_r cases, K and $K^\dagger$ annihilate all the states in V_B^M. Thus the quadratic Casimir operator takes the value $C_2 = M(M + 2r - 1)$ in the highest weight representation (6.4),(6.14).

One great difference between the D_r and B_r cases is the correspondence between the number states and weights. In the B_r case

$$|n_0, n_1, \ldots, n_r; \bar{n}_1, \ldots, \bar{n}_r\rangle \Rightarrow \mu = \sum_{j=1}^{r}(n_j - \bar{n}_j)e_j. \tag{6.16}$$

Namely, n_0, the number of a_0 quanta, has no effects on the weights.

The B_r coherent states can be constructed in a way similar to the D_r cases. The generators having non-trivial action on the highest weight states are

$$X_{(-1,j)}, \quad X_{(-1,-j)}, \quad 2 \le j \le r, \quad \text{and} \quad X_{(-1,0)}, \tag{6.17}$$

which commute among themselves, since the sum of the corresponding roots are no longer roots. They constitute one half of the generators corresponding to the quotient space

$$SO(2r+1)/U(1) \times SO(2r-1),$$

having the dimension

$$2(2r-1).$$

In terms of $2r-1$ complex parameters

$$\xi_0, \quad \xi_j, \quad \xi_{-j}, \quad 2 \le j \le r, \tag{6.18}$$

we define a linear combination of the non-trivial generators (6.17) as

$$T = \xi_0 X_{(-1,0)} + \sum_{j=2}^{r} \xi_j X_{(-1,j)} + \sum_{j=2}^{r} \xi_{-j} X_{(-1,-j)}. \tag{6.19}$$

Then the un-normalised coherent state is expressed as

$$\exp[T](a_1^\dagger)^M|0\rangle, \tag{6.20}$$

which leads, after repeated use of the formal Taylor theorem (3.9), to

$$\left(\xi_0 a_0^\dagger + a_1^\dagger + \sum_{j=2}^{r} \xi_j a_j^\dagger + \sum_{j=2}^{r} \xi_{-j} b_j^\dagger - (\frac{\xi_0^2}{2} + \sum_{j=2}^{r} \xi_j \xi_{-j}) b_1^\dagger \right)^M |0\rangle. \tag{6.21}$$

Thus we obtain the normalised B_r multinomial state

$$|\boldsymbol{\eta}; M; B_r\rangle = \frac{1}{\sqrt{M!}} \left(\eta_0 a_0^\dagger + \sum_{j=1}^{r} \eta_j a_j^\dagger + \sum_{j=1}^{r} \eta_{-j} b_j^\dagger \right)^M |0\rangle, \tag{6.22}$$

in which

$$\begin{aligned}
\eta_1 &= \left(1+\sum_{j=2}^{r}|\xi_j|^2+\sum_{j=2}^{r}|\xi_{-j}|^2+|\frac{\xi_0^2}{2}+\sum_{j=2}^{r}\xi_j\xi_{-j}|^2\right)^{-\frac{1}{2}}, \quad \eta_0=\xi_0\eta_1, \\
\eta_j &= \xi_j\eta_1,\ \eta_{-j}=\xi_{-j}\eta_1,\ 2\le j\le r,\ \eta_{-1}=-(\frac{\xi_0^2}{2}+\sum_{j=2}^{r}\xi_j\xi_{-j})\eta_1,
\end{aligned} \tag{6.23}$$

satisfying the condition

$$|\eta_0|^2+\sum_{j=1}^{r}(|\eta_j|^2+|\eta_{-j}|^2)=1.$$

Let us turn to the probability distribution. The B_r multinomial states give multinomial distribution to the *number states* with probabilities $|\eta_0|^2$, $|\eta_j|^2$ and $|\eta_{-j}|^2$:

$$\begin{aligned}
&|\langle n_0,n_1,\ldots,n_r;\bar{n}_1,\ldots,\bar{n}_r|\boldsymbol{\eta};M;B_r\rangle|^2 \\
&\quad=\frac{M!}{n_0!n_1!\cdots n_r!\bar{n}_1!\cdots\bar{n}_r!}|\eta_0|^{2n_0}|\eta_1|^{2n_1}\cdots|\eta_r|^{2n_r}|\eta_{-1}|^{2\bar{n}_1}\cdots|\eta_{-r}|^{2\bar{n}_r}.
\end{aligned} \tag{6.24}$$

By summing the contributions from all the number states belonging to a given weight $\boldsymbol{\mu}$ we obtain the B_r multinomial distribution:

$$\begin{aligned}
&B_{\boldsymbol{\mu}}(\boldsymbol{\eta};M) \\
&=\sum_{n_j-\bar{n}_j=\mu_j}\frac{M!}{n_0!n_1!\cdots n_r!\bar{n}_1!\cdots\bar{n}_r!}|\eta_0|^{2n_0}|\eta_1|^{2n_1}\cdots|\eta_r|^{2n_r}|\eta_{-1}|^{2\bar{n}_1}\cdots|\eta_{-r}|^{2\bar{n}_r}.
\end{aligned} \tag{6.25}$$

Here let us recall that n_0 has no effects on the weights. Thus the interpretation as "picking up coloured balls from a pot" is also valid but with a slight modification. In the pot we have $2r+1$ types of balls, among them r different colours and each colour has "positive" and "negative" types. There are also "colourless" (or "dummy") balls. They have probabilities $|\eta_j|^2$, $|\eta_{-j}|^2$ $(j=1,\ldots,r)$ and $|\eta_0|^2$. We pick up total of M balls and ask the probability distribution of the "net" number of coloured balls (or weights). It is given by the B_r multinomial distribution. As in the D_r multinomial distribution, among the probabilities $|\eta_0|^2$,$|\eta_1|^2$, $\ldots$, $|\eta_r|^2$, $|\eta_{-1}|^2$, $\ldots$, $|\eta_{-r}|^2$, only $2r-1$ of them are independent. As is clear from (6.23), one of the *dependent* probabilities, say $|\eta_{-1}|^2$, depends on the information of the other $\eta_{\pm j}$'s including their phases. The existence of the "colourless" balls (or dummy elements) and the "quantum" nature of η_{-1} are novel features of the B_r multinomial distributions.

7 Summary

Starting from the fact established in our previous work [14] that the coherent states of the Heisenberg-Weyl, $su(2)$, $su(r+1)$, $su(1,1)$ and $su(r,1)$ algebras in certain symmetric

(bosonic) representations give the well-known probability distributions, the Poisson, binomial, multinomial distributions with their “negative” counterparts, we have proceeded to the second stage in the study of “quantum probability”. By reversing the logic, we have obtained new probability distributions based on the coherent states of the classical algebras B_r, C_r and D_r in symmetric (bosonic) representations. These new probability distributions have similar features as the multinomial distributions related with A_r algebra. They also possess several new features reflecting their Lie algebraic and “quantum” backgrounds. As byproducts, simple proofs and interpretation of some addition theorems of Hermite polynomials are obtained based on the ‘coordinate’ representation of the (negative) multinomial states, the coherent states of $su(r+1)$ ($su(r,1)$) algebra in symmetric representations.

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Appendix Addition Theorems II

In this appendix we show a simple proof and interpretation of another type of addition theorems of Hermite polynomials. These theorems are non-compact counterparts of the theorems presented in section 3.3. They are obtained from the coordinate representation of the negative binomial and negative multinomial states, i.e., the coherent states of the $su(1,1)$ and $su(r,1)$ in symmetric representations. The theorem corresponding to the negative binomial states reads

$$\begin{aligned} & (1-\eta^2)^{-M/2} e^{x_0^2 - \frac{(x_0-\eta x_1)^2}{1-\eta^2}} H_{M-1}\left(\frac{x_0-\eta x_1}{\sqrt{1-\eta^2}}\right) \\ = & \sum_{n=0}^{\infty} \frac{(\eta/2)^n}{n!} H_{n+M-1}(x_0) H_n(x_1), \end{aligned} \tag{A.1}$$

in which η is a complex parameter $|\eta| < 1$. This addition theorem is known as generalised Mehler formula [22, 23] but is not found in the standard mathematics reference texts, except for the simplest case with $M = 1$ which is well-known as Mehler formula (see, for example, p194 of [19]). For a detailed characterisation of the negative binomial (multinomial) distributions in terms of Lie algebras, we refer to our previous work [14].

Let us begin with the negative binomial distribution (here $\eta \in \mathbf{R}$ for simplicity):

$$B_n^-(\eta; M) = \binom{M+n-1}{n} \eta^{2n}(1-\eta^2)^M, \quad n = 0, 1, \ldots, \tag{A.2}$$

which describes the probability distribution of the "waiting time" [20]. Suppose we play Bernoulli's trial of success and failure in which the probability of *failure* is $0 < \eta^2 < 1$. The probability distribution for n, such that the (preset) M-th ($M \geq 1$, integer) success turns out at the $M+n$-th trial, is given by the above formula (A.2). We follow the examples of the previous sections and construct the "probability amplitude" of the negative binomial distribution. We choose the following restricted bosonic Fock space built by two bosonic oscillators:

$$\begin{aligned}
[a_j, a_k^\dagger] &= \delta_{jk}, \quad a_j|0\rangle = 0, \quad j, k = 0, 1, \\
|n_0; n_1\rangle &= \frac{a_0^{\dagger n_0} a_1^{\dagger n_1}}{\sqrt{n_0! n_1!}}|0\rangle, \quad n_0 - n_1 = M - 1, \quad n \geq 0.
\end{aligned} \tag{A.3}$$

Here n_0 is the total number of trials except for the final one and n_1 is the number of failures (the final trial is always a success, by definition). Obviously this Fock space is infinite dimensional. We look for a state $|\eta; M\rangle^-$ such that

$$|\langle n_0; n_1|\eta; M\rangle^-|^2 = B_{n_1}^-(\eta; M).$$

For a special choice of the phases (cf. (2.5)) we arrive at a very simple result

$$\begin{aligned}
|\eta; M\rangle^- &= \sum |n_0; n_1\rangle\langle n_0; n_1|\eta; M\rangle^- \\
&= (1-\eta^2)^{\frac{M}{2}} \sum |n_0; n_1\rangle \eta^n \sqrt{\frac{n_0!}{n_1!(M-1)!}} \\
&= (1-\eta^2)^{\frac{M}{2}} \sum_{n_1=0}^{\infty} \frac{(\eta a_0^\dagger a_1^\dagger)^{n_1}}{n_1!} \frac{(a_0^\dagger)^{M-1}}{\sqrt{(M-1)!}}|0\rangle \\
&= (1-\eta^2)^{\frac{M}{2}} e^{\eta a_0^\dagger a_1^\dagger}|M-1; 0\rangle.
\end{aligned} \tag{A.4}$$

This is called the negative binomial state [13, 14]. This is exactly an $su(1,1)$ coherent state as we will see presently. The $su(1,1)$ algebra is realised in the above Fock space as

$$\begin{aligned}
K_+ &= a_0^\dagger a_1^\dagger, \quad K_- = a_0 a_1, \quad K_0 = \frac{1}{2}(N_0 + N_1 + 1), \quad N_j = a_j^\dagger a_j, \\
[K_+, K_-] &= -2K_0, \quad [K_0, K_\pm] = \pm K_\pm.
\end{aligned} \tag{A.5}$$

The lowest weight state is $|M-1; 0\rangle$:

$$K_-|M-1; 0\rangle = 0, \quad K_0|M-1; 0\rangle = \frac{M}{2}|M-1; 0\rangle, \tag{A.6}$$

which gives rise to the discrete irreducible representation with Bargman index $M/2$. Thus the un-normalised coherent state is ($\eta \in \mathbf{C}$)

$$e^{\eta K_+}|M-1;0\rangle = e^{\eta a_0^\dagger a_1^\dagger}|M-1;0\rangle, \tag{A.7}$$

which has the same form as given in (A.4).

Next we take the coordinate representation of the above negative binomial state:

$$\langle x_0; x_1|e^{\eta a_0^\dagger a_1^\dagger}|M-1;0\rangle$$

and evaluate it in two different ways. The first is to simply expand the exponential and use the formula (3.33):

$$\langle x_0; x_1|e^{\eta a_0^\dagger a_1^\dagger}|M-1;0\rangle = \frac{e^{-\frac{1}{2}(x_0^2+x_1^2)}}{\pi^{1/2}\sqrt{(M-1)!}}\sum_{n=0}^{\infty}\frac{(\eta/2)^n}{n!}H_{n+M-1}(x_0)H_n(x_1), \tag{A.8}$$

which corresponds to the right hand side of (A.1).

The second is to use the coordinate representation of the creation operators

$$a_j^\dagger = \frac{1}{\sqrt{2}}(x_j - \frac{\partial}{\partial x_j}) = -\frac{1}{\sqrt{2}}e^{\frac{1}{2}x_j^2}D_j e^{-\frac{1}{2}x_j^2}, \quad D_j = \frac{\partial}{\partial x_j}, \quad j = 0, 1,$$

to obtain

$$\langle x_0; x_1|e^{\eta a_0^\dagger a_1^\dagger}|M-1;0\rangle = \frac{(-1)^{M-1}}{\pi^{1/2}\sqrt{(M-1)!}}\, e^{\frac{1}{2}(x_0^2+x_1^2)}e^{\eta D_0 D_1/2}D_0^{M-1}e^{-(x_0^2+x_1^2)}.$$

By applying the formal Taylor theorem (3.9) with respect to x_1 by treating ηD_0 as a parameter, we obtain

$$\begin{aligned}
&\langle x_0; x_1|e^{\eta a_0^\dagger a_1^\dagger}|M-1;0\rangle \\
=\;& \frac{(-1)^{M-1}e^{\frac{1}{2}(x_0^2+x_1^2)}}{\pi^{1/2}\sqrt{(M-1)!}}\, D_0^{M-1}e^{-(x_1+\eta D_0/2)^2}e^{-x_0^2} \\
=\;& \frac{(-1)^{M-1}e^{\frac{1}{2}(x_0^2-x_1^2)}}{\pi^{1/2}\sqrt{(M-1)!}}\frac{1}{\sqrt{1-\eta^2}}\, e^{-\eta x_1 D_0}D_0^{M-1}e^{-\frac{x_0^2}{1-\eta^2}},
\end{aligned} \tag{A.9}$$

which gives a scaled ($1/\sqrt{1-\eta^2}$) and shifted ($-\eta x_1$) Hermite polynomial (H_{M-1}) by Rodrigues formula (3.34):

$$\text{r.h.s of (A.9)} = \frac{1}{\pi^{1/2}\sqrt{(M-1)!}}(1-\eta^2)^{-\frac{M}{2}}e^{\frac{1}{2}x_0^2}e^{-\frac{(x_0-\eta x_1)^2}{1-\eta^2}}H_{M-1}\left(\frac{x_0-\eta x_1}{\sqrt{1-\eta^2}}\right). \tag{A.10}$$

Here use is made of a simple formula

$$e^{tD_0^2}e^{-x_0^2} = \frac{1}{\sqrt{1+4t}}e^{-\frac{x_0^2}{1+4t}}, \quad |t| < \frac{1}{2},$$

which can be proved, for example, by taking the Fourier transform. By comparing (A.9) and (A.10) we arrive at the addition theorem of Hermite polynomials given above (A.1). It should be remarked that the generalised Mehler formula (A.1) is also obtained from Mehler formula ($M = 1$) by differentiating with respect to x_0 $M-1$ times.

Generalisation to the negative multinomial distribution

$$\begin{aligned} M^-_{\mathbf{n}}(\boldsymbol{\eta}\,;M) &= (1-\boldsymbol{\eta}^2)^M \frac{(M+n_1+\cdots+n_r-1)!}{\mathbf{n}!(M-1)!}\eta_1^{2n_1}\cdots\eta_r^{2n_r}, && \text{(A.11)}\\ \mathbf{n} &= (n_0,n_1,\ldots,n_r), \qquad \boldsymbol{\eta}=(\eta_1,\ldots,\eta_r)\in\mathbf{R}^r, && \text{(A.12)}\\ 0 &< \boldsymbol{\eta}^2 = \eta_1^2+\cdots+\eta_r^2 < 1, \end{aligned}$$

is rather straightforward. We introduce a restricted Fock space generated by $r+1$ oscillators:

$$\begin{aligned} [a_j, a_k^\dagger] &= \delta_{jk}, \quad a_j|0\rangle = 0, \quad j=0,1,\ldots,r, && \text{(A.13)}\\ |n_0;n_1,\ldots,n_r\rangle &= \frac{(a_0^\dagger)^{n_0}(a_1^\dagger)^{n_1}\cdots(a_r^\dagger)^{n_r}}{\sqrt{n_0!n_1!\cdots n_r!}}|0\rangle, \quad n_0-(n_1+\cdots+n_r)=M-1. \end{aligned}$$

Then the "square root" of the negative multinomial distribution is

$$|\boldsymbol{\eta}\,;M\rangle^- = (1-\boldsymbol{\eta}^2)^{\frac{M}{2}} e^{a_0^\dagger(\sum_{j=1}^r \eta_j a_j^\dagger)}|M-1;0,\ldots,0\rangle, \tag{A.14}$$

which is an $su(r,1)$ coherent state in an irreducible symmetric representation with the lowest weight state

$$|M-1;0,\ldots,0\rangle. \tag{A.15}$$

The generators are

$$\begin{aligned} K_{+j} &= a_0^\dagger a_j^\dagger, \quad K_{-k} = a_0 a_k, \quad 1\le j,k\le r,\\ K_{jk} &= a_j^\dagger a_k \quad (j\ne k\ne 0), \quad N_j = a_j^\dagger a_j. && \text{(A.16)} \end{aligned}$$

It is easy to see that they leave the combination

$$\Delta \equiv N_0 - (N_1+\cdots+N_r)$$

and the above Fock space (A.13) invariant. Among the above generators the following r generators have non-trivial action on the lowest weight state (A.15)

$$K_{+j} = a_0^\dagger a_j^\dagger, \quad j=1,\ldots,r. \tag{A.17}$$

Thus in terms of r complex parameters $\eta_1,\ldots,\eta_r$, satisfying the condition

$$|\boldsymbol{\eta}|^2 = \sum_{j=1}^r |\eta_j|^2 < 1, \tag{A.18}$$

we obtain an un-normalised negative multinomial state

$$e^{\sum_{j=1}^{r}\eta_j K_{+j}}|M-1;0,\ldots,0\rangle = e^{a_0^\dagger(\sum_{j=1}^{r}\eta_j a_j^\dagger)}|M-1;0,\ldots,0\rangle, \tag{A.19}$$

which has the same form as (A.14). By evaluating the coordinate representation of the above state (A.19) in two different ways, we obtain another form of addition theorem of Hermite polynomials:

$$\begin{aligned}&(1-\boldsymbol{\eta}^2)^{-M/2}e^{x_0^2-\frac{(x_0-\eta_1x_1-\cdots-\eta_rx_r)^2}{1-\eta_1^2\cdots-\eta_r^2}}H_{M-1}\left(\frac{x_0-\eta_1x_1-\cdots-\eta_rx_r}{\sqrt{1-\eta_1^2\cdots-\eta_r^2}}\right)\\ =&\sum_{n_j=0}^{\infty}\frac{(\eta_1/2)^{n_1}}{n_1!}\cdots\frac{(\eta_r/2)^{n_r}}{n_r!}H_{M+n_1\cdots+n_r-1}(x_0)H_{n_1}(x_1)\cdots H_{n_r}(x_r),\end{aligned} \tag{A.20}$$

One can obtain this addition theorem by combining the addition theorems from the multinomial state (3.32) and that of the negative binomial state (A.1), which reflects the fact that the negative multinomial state is also obtained by combining the negative binomial state and the multinomial state.

Before closing Appendix, let us mention another interesting form of addition theorems of Hermite polynomials which is obtained as a special case of (A.1). By setting $x_0 \equiv x$ and $x_1 \equiv 0$, we obtain

$$(1-\eta^2)^{-M/2}e^{-\frac{\eta^2}{1-\eta^2}x^2}H_{M-1}(\frac{x}{\sqrt{1-\eta^2}}) = \sum_{n=0}^{\infty}\frac{(-\eta^2/4)^n}{n!}H_{2n+M-1}(x). \tag{A.21}$$

Here use is made of the relations

$$H_{2n}(0) = (-1)^n(2n-1)!! = (-1)^n 1\cdot 3\cdots(2n-1), \quad H_{2n+1}(0) = 0.$$

This form of addition theorems can also be obtained from another type of "coherent states" of $su(1,1)$ algebra. Let us take the single boson Fock space (2.6)–(2.8) with the basis $\{|n\rangle,\ n=0,1,\ldots,\}$ generated by a and $a^\dagger$. The $su(1,1)$ algebra is realised by

$$K_+ = \frac{1}{2}(a^\dagger)^2, \quad K_- = \frac{1}{2}a^2, \quad K_0 = \frac{1}{2}a^\dagger a + \frac{1}{4}. \tag{A.22}$$

As before evaluate an un-normalised "coherent state"

$$e^{tK_+}|M-1\rangle = e^{\frac{t}{2}(a^\dagger)^2}|M-1\rangle, \quad |t|<1, \tag{A.23}$$

in two different ways ($t=-\eta^2$). The above state is known as the 'squeezed number state' in quantum optics [21], for the 'base state' $|M-1\rangle$ is not of lowest weight.

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